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# The application of 2D and 3D graphic statics in design

C. HARTZ*, A. MAZUREK[a], M. MIKI[a], T. ZEGARD[b], T. MITCHELL[a], W. F. BAKER[a]

* Skidmore, Owings and Merrill LLP
224 S. Michigan Ave. 1000, Chicago, IL, 60604, US
christian.hartz@som.com

[a] Skidmore, Owings and Merrill LLP
[b] Pontificia Universidad Católica de Chile, Chile

## Abstract

This paper focuses on the use of graphic statics in both the two and three-dimensional layout of structures, and in the purely geometric panelization of architectural surfaces.

The paper will present an inhouse computer program for graphic statics including an untangling algorithm, a topology optimizer based on the Newton method with constraints and discrete step as well as an optimization procedure based on virtual work.

In the second section, we show how graphic statics can be generalized to 3D to enable form-finding of 3D funicular structures. In contrast to other form-finding techniques like the force density method, this method is straightforward to use in the design of structures that have both compression and tension forces. We demonstrate the utility of this approach via examples of form-finding of a bridge with a tensioned curved deck supported by a spatially curved compression arch.

Finally, we will show how graphic statics can be used in a purely geometric context to lay out flat panels on architectural surfaces of arbitrary topological complexity. Since graphic statics relates the equilibrium of forces in a 2D truss structure to the orthogonal projection of a 3D plane-faced polyhedron (the discrete Airy function), it is possible to use 2D equilibrium to define and control 3D plane-faced architectural surfaces. This approach provides an intuitive method for engineers, normally more familiar with equilibrium than pure geometry, to understand and rationalize architectural surfaces. Using this approach, we show how graphic statics can be used to derive, explore, and reason about architectural planar panelizations. Unlike variational approaches to surface planarization, it is straightforward to handle a surface of any topology; pentagons, octagons etc. are treated just as easily as quads and triangles. We also believe this approach allows clearer reasoning about the limitations and design degrees-of-freedom of different panelization strategies.

**Keywords**: dual structures, graphic statics, optimization, planarization, architectural geometry, form finding, reciprocal diagrams, topology optimization

## 1. Graphic statics optimizer

### 1.1. Implementation

It has been assumed in this chapter that the reader knows graphic statics (Zalewski and Allen [1], Baker et. al. [2], Beghini et. al. [3], Mazurek et. al. [4]) and graph theory (Bondy and Murty [5] especially chapter 10 on Planar Graphs).

Members of a truss structure (*form diagram*) are understood here to compose a single connected planar graph. In graphic statics, the first task an engineer needs to do is to annotate all nodes of the truss and polygons between the members of the truss. It shall be noted here that the graphic statics method in its

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classic form is valid only for planar graphs where each edge is adjacent to 2 different polygons and where no 2 edges have the same 2 polygons adjacent to them. That is, the diagram must be a single sheet; multiple sheets cannot be joined. Another important restriction, that needs to be noted, is that only nodes interconnected with members (edges) that are adjacent to one polygon can receive a load or reaction. This single polygon is called external or *loaded polygon*.

To set up dual form and force diagrams, it is first necessary to check that the form diagram is a planar graph. Finding planarity of a graph in general is not an easy task. Coding the process in the computer is even more difficult. An algorithm on how to find efficiently planarity of a graph has been described in Hopcroft and Tarjan [6].

Notation is a very important element in keeping graphic statics diagrams simple and consistent. Notation used to develop the **G**raphic **S**tatics **O**ptimizer (GSO) is named Bow's notation (Bow [7], Zalewski and Allen [1]). In this notation, polygons in the form diagram are annotated with numbers: 1, 2, 3, … and capital letters: A, B, C, … (see Figure 1). All open polygons marked with letters are parts of a single closed *loaded polygon* separated with loaded nodes. Polygons in *form diagram* correspond (are reciprocal to) nodes in *force diagram*. These reciprocal nodes are annotated with numbers and small letters that correspond to polygons they are reciprocal to: 1, 2, 3, … , a, b, c, … .

Edges in *the form diagram* are reciprocal to lines in *the force diagram*. These lines are parallel to each other and lengths of the lines in *the force diagram* are proportional to the forces in the structural edges shown in *the form diagram*.

The graphical procedure of creating *a force diagram* for a given *form diagram* by drawing parallel lines can be described mathematically with a set of linear equations. Solving these equations, a computational algorithm is able to locate all nodes of *the force diagram*. These equations are:

- Equation describing lines of forces (in *force diagram*) to be parallel to corresponding lines of edges (in *form diagram*) – 1 equation per edge is:

$$(y_2 - y_1)(x'_2 - x'_1) = (x_2 - x_1)(y'_2 - y'_1) \tag{1}$$

  Where $x$, $y$ are coordinates of nodes, indexes 1 and 2 indicate start and end node of a line and prime sign indicates node in the *force diagram*.

- Equations describing distance between 2 nodes in the *force diagram* that are reciprocal to 2 adjacent *loaded polygons* to be equal to the force applied at the node – 2 equations per loaded node are:

$$(x'_2 - x'_1) = F_x \tag{2}$$

$$(y'_2 - y'_1) = F_y \tag{3}$$

  Where $F_x$, $F_y$ are components of a vector representing nodal load.

Example shown in Figure 1 has 8 edges and 1 loaded node = 10 equations. Number of unknown coordinates of nodes reciprocal to the polygons is 2 x 6 = 12. However, one of the reciprocal nodes (2 unknowns) can be located freely in a 2D space.

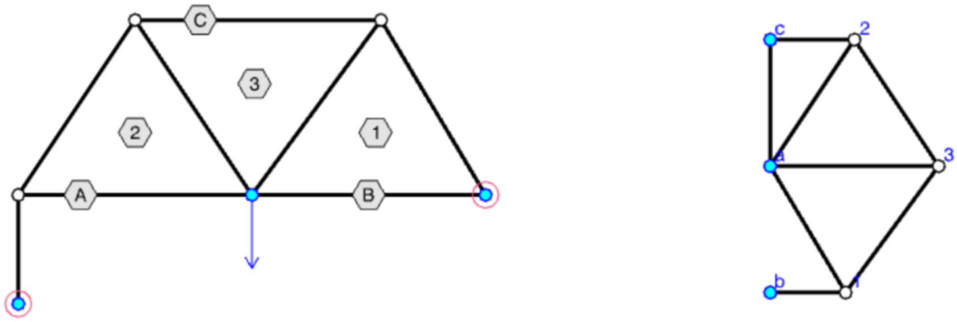



Figure 1: Form and force diagrams of a truss obtained using GSO tool

Note that support reactions are not part of the linear equations above. Because supports are assumed to restrain nodes in both directions the reaction equations are not required.

### 1.2. Benefits of the graphic statics

The *graphics statics* method was used to solve trussed structures on a daily basis before the development of computers. Today solving trusses is trivial using any basic finite element (FE) software. Thus, *graphics statics* has been largely forgotten. However, there are other benefits to the method besides obtaining structural forces. An obvious one is that the magnitude of member forces is represented graphically and it is easy to notice when a force becomes large as the graphics will show a long line unlike color scale solutions of FE programs.

Another benefit of this method is that an engineer may want to modify member forces in the *force diagram* and obtain a structure required to result in such forces. This is called *design in the force domain*. Figure 2 shows Robert Maillard truss (1924). This roof truss has been designed to have constant force in the top chord of the truss (constant lengths of lines c-6, d-5, e-3, f-2, g-1 and a-4) and to have zero force in the diagonal web members. Note that diagonal members (dashed lines) have been removed from the model after the geometry has been found making the system statically unstable. Nevertheless, this method still provides statically correct solution.

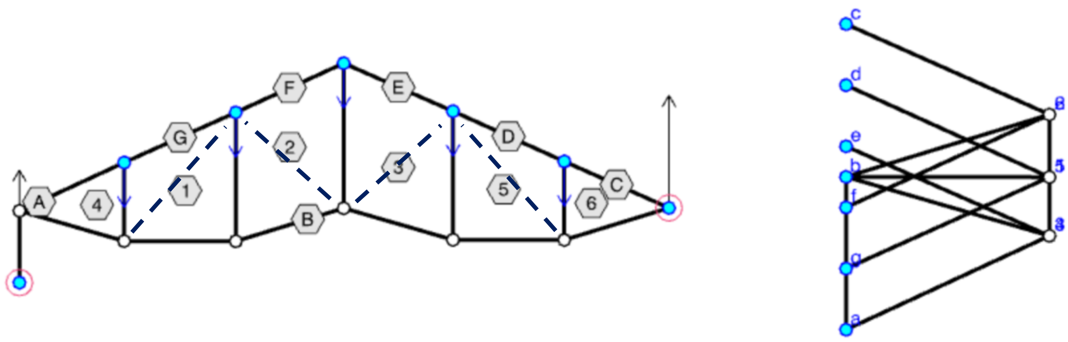



Figure 2: Form and force diagrams of a Robert Maillard truss obtained using GSO tool

The graphics statics tool can also be used to find the stiffest truss within a described connectivity. The stiffest truss is that with the minimal *load path* (Baker et. al. [2]) defined with (4).

$$\delta = \sum |f| \cdot l \tag{4}$$

Where: $f$ are member forces and $l$ are member lengths.

Value of *load path* is proportional to minimum structural quantities for structure with uniform absolute stress not exceeding a stress limit, i.e., plastic design. At the same time *load path* is proportional to total stiffness / elastic energy of a structure with fixed volume (Mazurek et al. [8]). Minimizing the *load path* is therefore beneficial for both minimizing structural quantities and deflections, since, for a single load case, one problem can be transformed into the other.

The graphics statics makes the calculation of the *load path* much easier. The summation in (4) is the dot product of the lengths of each line from the two diagrams.

### 1.4. Dual structures

*Form* and *force diagrams* are reciprocal to each other. This means that the geometric rules used to obtain a *force diagram* from a *form diagram* can be used on the *force diagram* to generate back the original *form diagram*. It can be proven that any *force diagram*, viewed as lines of a structure, can be loaded in such a way that lines of the original *form diagram* are its *force diagram*. Such two structures are called *dual structures*. Both *dual structures* have the same *load path*. If one of these structures is optimal (has minimum *load path*) the other is also optimal for its set of forces. An example of 2 optimal dual structures is shown in Figure 3. More optimal *dual structures* are presented in Baker et. al. [2].

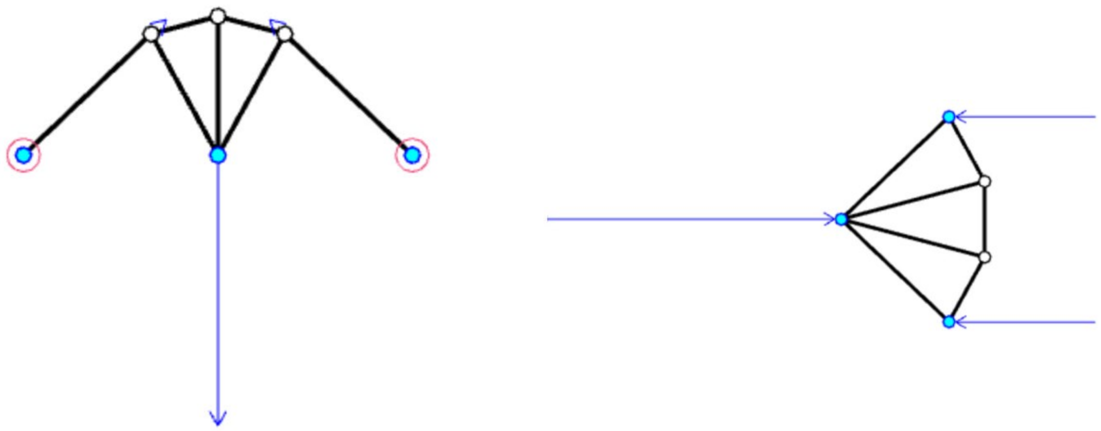


Figure 3: Equally optimal *dual structures* obtained using GSO tool

### 1.5. Optimization

Modifying the above solver by adding an extension to automatically find efficient structures was an easy addition. After defining a structural model (topology, loads and constrains) the user needs to define which nodes can be relocated and in what directions. The program will automatically find most efficient locations for these nodes. Currently 2 minimization methods are implemented: gradient based – Newton method with constraints (fast) and discrete – coordinate descent method (slow) (Nocedal and Wright [9]). The discrete method is used when the gradient based method locks in a local minimum. The truss in Figure 4 is an outcome of optimization of a simply supported spanning truss. Note that this truss is unstable without moment continuity of the top and bottom chords. Nevertheless, the *force diagram* shown in the bottom of the same figure is statically correct.

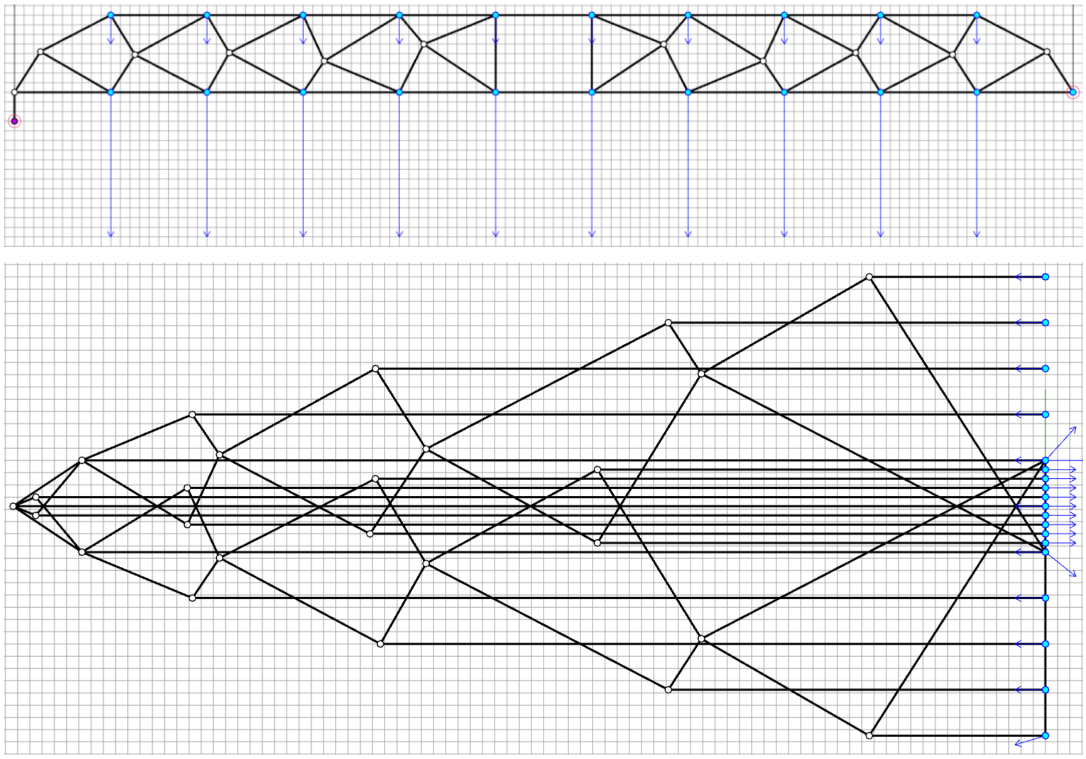


Figure 4: Example of optimized truss using GSO tool

## 2. 3D graphic statics applied to a pedestrian bridge design

Graphic statics is conducted on a flat plane. However, because the essence of graphic statics is to have all lines between form and force diagrams parallel and to close all the polygons in the force diagram, both form and force diagrams can, in principle, take three-dimensional forms.

Figure 5 and Figure 6 show a design example of a pedestrian bridge that used three-dimensional graphic statics. The bridge consists of one complete arch and two semi-arches as noted in Figure 5.

Figure 7 shows the 3D form and force diagrams constructed for designing this pedestrian bridge. By considering the symmetry of the bridge, it is possible to complete the analysis of the entire structure. In the force diagram, the series of deck lines (red) and the series of arch lines (purple) seem to be flipped around the strut lines (blue). This indicates that if the arch is in compression, the deck is in tension.

The major advantage of the use of graphics statics is that it gives the designer direct control of the design and does not require any computationally expensive nonlinear optimization technique. As illustrated in Figure 8, the construction of the form and force diagram shown in Figure 7 is straightforward. It was decided to make the bridge symmetric about the mid-span. Because the semi-arch is supposed to be connected to its symmetric pair, we assume a mirror plane at the one end of the semi-arch. The first deck line does not need to be perpendicular to the mirror plane but the reaction force does. However, because the magnitude of the reaction force is unknown, it is treated as the first variable parameter. Then the direction of the first strut can be identified by computing an intersection of the mirror plane and the first deck line in the force diagram. Because this only determines the direction of the first strut, the length of the first strut is treated as the second variable parameter. The second reaction force acting on the bottom end of the strut also needs to be perpendicular to the mirror plane but its magnitude is also unknown. This is treated as the third variable parameter. Other than those three parameters, the curve of deck lines is also a variable parameter. We decided to use a circular arc divided by a consistent length as the deck lines. Hence, the parameters now are the directions of the both ends of the deck, total length of the deck, and the three parameters mentioned above. When designing this bridge, one needs to adjust those parameters so that the overall shape has a decent aesthetics and all the arches touch the ground.

After constructed those initial elements, rest of the process is repetitions of the same task. Because lines are parallel between form and force diagrams, planes spanned by two lines should also be parallel between two diagrams. Using this, length of the next deck line in the force diagram can be identified by computing an intersection between a plane and a line. Then the direction of the next strut can be identified. Repeating this, the entire form of the bridge will emerge.

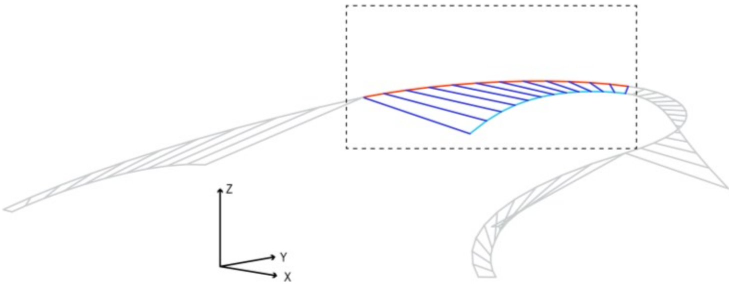
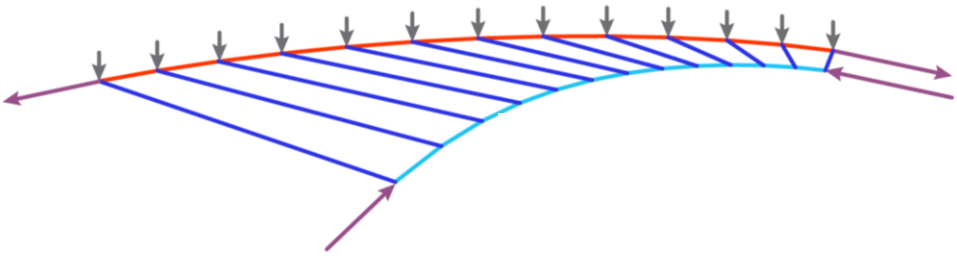
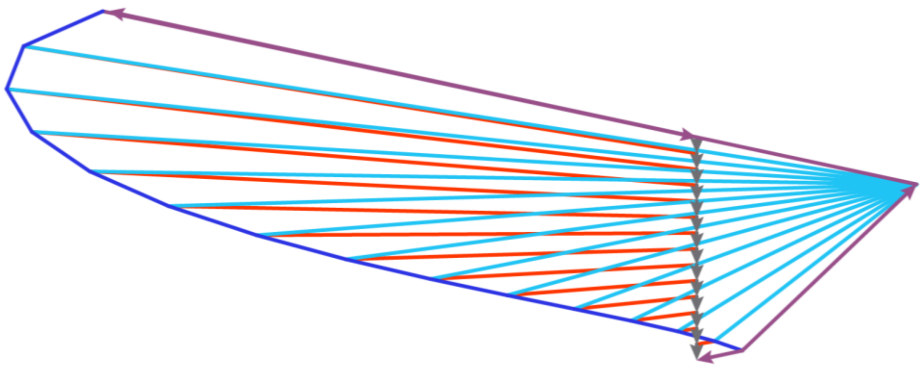



Figure 5: A design example of pedestrian bridge using three-dimensional graphic statics

Figure 6: A rendered image of the bridge placed in a context



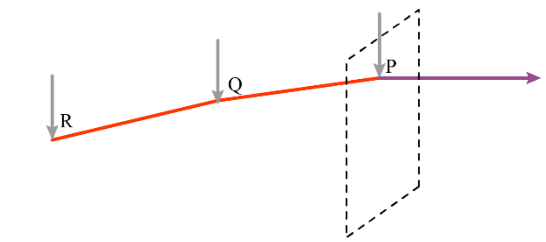
(a) Form diagram, applied loads (grey) and reaction forces (purple) – Perspective view



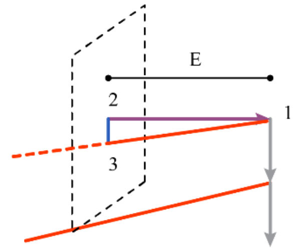
(b) Three-dimensional force diagram for Figure 5 – Perspective view



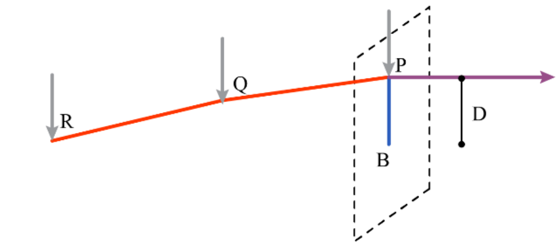
Figure 7: Constructed form (a) and force (b) diagrams for highlighted section in Figure 5



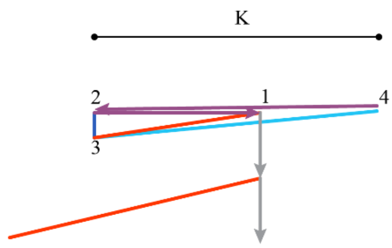

(a) Form diagram: reaction force (purple) should be perpendicular to the plane that deterrmines mirror symmetry.



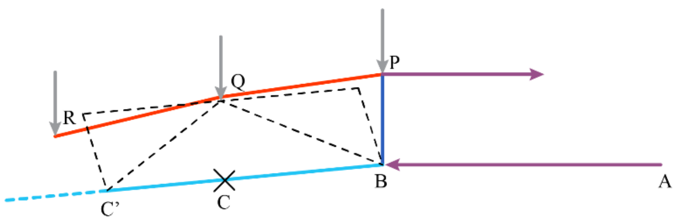

(b) Force diagram: because the magnitude of the reaction force is unknown, it is treated as a variable parameter (E). By computing an intersection between the plane and the first red line, point 3 can be identified.



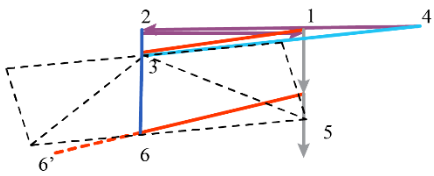

(c) Form diagram: while the direction of the first strut is determined (blue), the length is unknown. So the length of the first strut is also treated as a variable parameter (D).



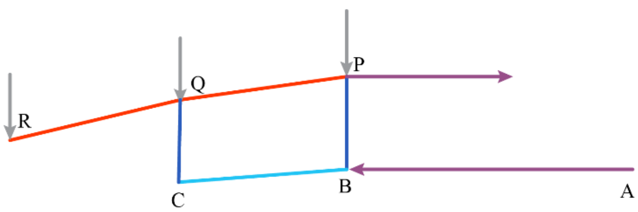

(d) Force diagram: since the magnitude of the reaction force acting on point B is also unknown, it is also treated as a variable parameter (K). Point 4 is determined and we obtain line 3-4.




(e) Form diagram: line B-C should be parallel to line 3-4 in the force diagram. To identify the location of C we look at a plane spanned by B, C' and Q.




(f) Force diagram: because the direction of the second deck line is known, point 6 can be identified by computing the plane in (e) and line 5-6'.




(g) Form diagram: line Q-C is known and parallel to line 3-6 in (f). Point C can be identified.

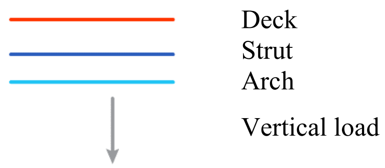



Figure 8: Construction of form (Figure 7a) and force (Figure 7b) diagrams

## 3. Graphic statics and panelization

Although, the theory of graphic statics was developed as a tool to find a structurally rational form, there is also an interesting application of the theory as be used in a purely geometric context.

It has been pointed out that given a form diagram, when a force diagram exists, it is possible to construct a planar faced polyhedron whose projection is exactly the form diagram. Many authors, for example Fraternali [10], have pointed out that this polyhedron is a 'discrete' version of the Airy stress function, since on an Airy stress function second derivatives represent stress tensors whereas on the plane faced polyhedron the 'jump' of first derivatives over across edges between adjacent faces represent axial forces of a linear members in the form diagram. The connection to the Airy function was proven rigorously in Mitchell et al [11]. The existence of such a plane faced polyhedron is not obvious for arbitrary form diagrams, but once a form diagram is adjusted such that a force diagram truly exists, this plane faced polyhedron always exists regardless of whatever each polygonal face is. It is truly surprising that even heptagon, octagon, or enneagon can satisfy planarity conditions in a precise sense.

This polyhedron has a high potential to be used for architectural designs, since even though construction technology is rapidly evolving and glass panels have some capacity to undergo deformation when being installed to a building, flat glass panels are preferable. Often complex surfaces are represented by triangular panels. Triangular panelization leads to very complicated connections. Flat faced polygonal surfaces tend to have fewer edges connected at a single point and therefore result in significantly simpler connections.

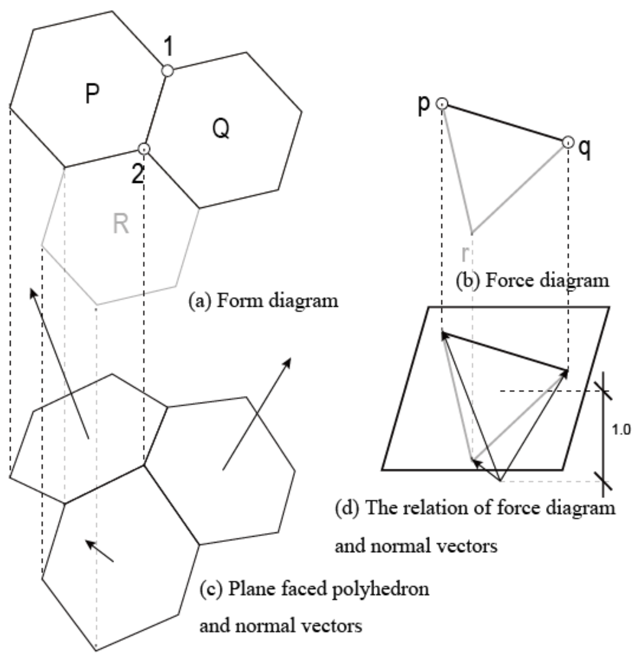



Figure 9: Relation between (a) form diagram, (b) force diagram, (c) planar faced polyhedron and (d) normal vectors of faces in the planar faced polyhedron

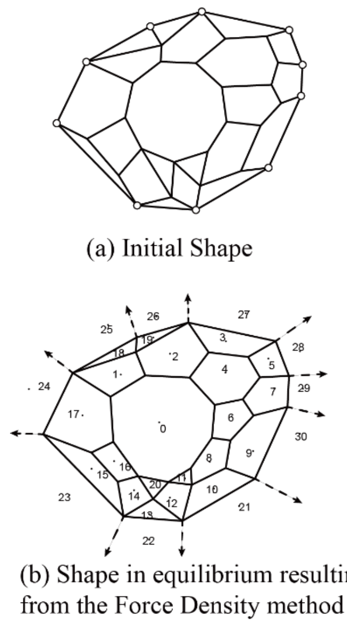


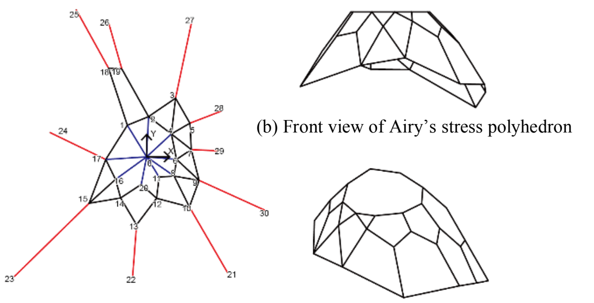

Figure 10: Force Density Method

Figure 9 shows the relation of planar faced polyhedron and its normal vectors to a pair of form and force diagrams. While form diagram represents the horizontal projection of the polyhedron, the force diagram represents the x and y components of the vectors normal to faces of the polyhedron. The normal vectors are 'normalized' for their z component equal 1.0. In other words, the force diagram has a degree of freedom of parallel translation, as well as the polyhedron and the normalized vectors.

A simple method to recover this polyhedron would be using the Force Density method [12]. Figure 10 shows an arbitrary layout pretensioned by a set of external forces. The plane faced polyhedron (b, c) can be obtained by solving a linear system of equations.

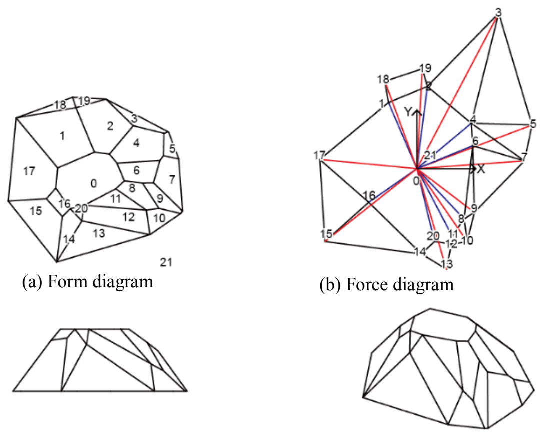
Note that, the boundary of the obtained polyhedron is arbitrary and does not lie on a plane. The geometrical requirement of planarity of the base can be interpreted in the force diagram by all external forces intersecting at a single point. External forces may be replaced with members connected at a single point and the resulting structure prestressed with a non zero force (see red lines in (b) of Figure 12). The point where all external loads or prestressing members meet defines the vector normal to the base. If the base is required to be horizontal the point needs to be located at (x,y) = (0,0). For engineer or architect, in order to find a form and force diagram, it is recommended to use a software that supports parametric constraints such as GSO discussed above or CATIA. Diagrams in Figure 12 were constructed using such software. The recovered polyhedron has horizontal planar polygons for bottom and top faces.

Figure 13 shows some renders of planar faced polyhedra that were designed by graphic statics. These models have potential to be used as architectural designs in the future.



(a) Force diagram (b) Front view of Airy's stress polyhedron (c) Isometric view of Airy's stress polyhedron

Figure 11: graphic statics and recovery of Airy's stress polyhedron which started with a Force Density method



(a) Form diagram (b) Force diagram (c) Front view of Airy's stress polyhedron (d) Isometric view of Airy's stress polyhedron

Figure 12: graphic statics of a self-equilibrium structure and recovery of Airy's stress polyhedron

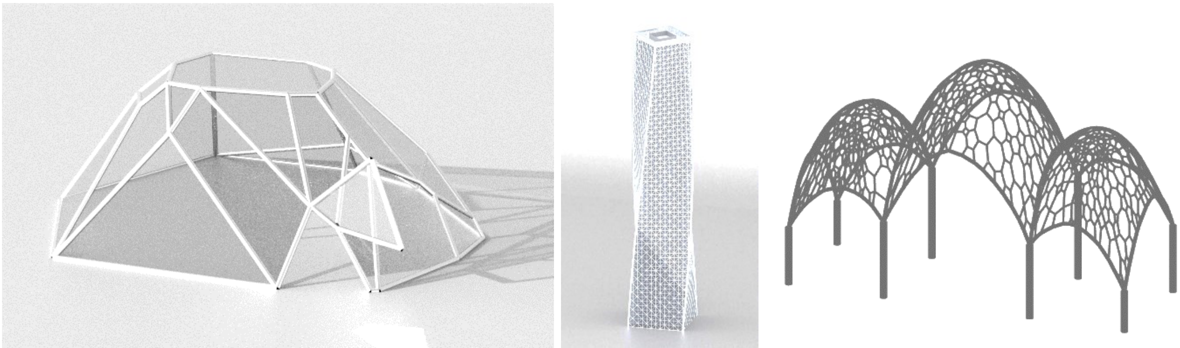


Figure 13: Application of techniques discussed in this section.

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